2021-03-26 updated

$\varphi \in L^2(G(k)\backslash G(\mathbb{A}), \omega)$

$x \mapsto \varphi\left(\begin{pmatrix} 1 & x \\ & 1 \end{pmatrix} g\right)$ function on $\mathbb{A}$ invariant by $k$ $L^2(k\backslash\mathbb{A})$

$\mathbb{A}/k \sim (\mathbb{R}/\mathbb{Z})^r$

Fix $\tau$ non-triv. add. char. $\mathbb{A}/k \to \mathbb{C}^1$

( $\tau(\xi\,\cdot)$ gives all characters of $\mathbb{A}/k$ as $\xi$ runs over $k$.)

$$\varphi\left(\begin{pmatrix} 1 & x \\ & 1 \end{pmatrix} g\right) = \sum_{\xi\in k} \varphi_\xi(g)\,\tau(\xi x)$$

$$\varphi_\xi(g) := \int_{\mathbb{A}/k} \varphi\left(\begin{pmatrix} 1 & x \\ & 1 \end{pmatrix} g\right) \overline{\tau(\xi x)}\, dx$$

normalise $dx$ $\mathrm{vol}(\mathbb{A}/k) = 1$.

For $\xi \neq 0$ $\quad \varphi_\xi(g) = \displaystyle\int_{\mathbb{A}/k} \varphi\left(\boxed{\begin{pmatrix} 1 & \xi^{-1}x \\ & 1 \end{pmatrix}} g\right)\overline{\tau(x)}\, dx = W_\varphi\left(\begin{pmatrix} \xi & \\ & 1 \end{pmatrix} g\right)$

$x \mapsto \xi^{-1}x$

$$\varphi\left(\underbrace{\cancel{\begin{pmatrix} \xi^{-1} & \\ & 1 \end{pmatrix}}}_{\in G(k)}\begin{pmatrix} 1 & x \\ & 1 \end{pmatrix}\begin{pmatrix} \xi & \\ & 1 \end{pmatrix}\right)$$

Set $\displaystyle W_\varphi(g) := \int_{\mathbb{A}/k} \varphi\left(\begin{pmatrix} 1 & x \\ & 1 \end{pmatrix} g\right)\overline{\tau(x)}\,dx$

global Whittaker function

$$\varphi(g) = \varphi_0(g) + \sum_{\xi \in k^\times} W_\varphi\left(\begin{pmatrix}\xi & \\ & 1\end{pmatrix} g\right)$$

<u>Defn</u>: $\varphi$ is a <u>cuspidal function</u> if the constant term $\varphi_0 = 0$

i.e., $\displaystyle\int_{\mathbb{A}/k} \varphi\left(\begin{pmatrix}1 & x \\ & 1\end{pmatrix} g\right) dx = 0$ for $g \in G(\mathbb{A})$ a.e.

$$\varphi \in \underbrace{L_0^2(G(k)\backslash G(\mathbb{A}), \omega)}_{\substack{\text{space of cuspidal functions} \\ \text{closed invariant subspace}}} \subseteq L^2(G(k)\backslash G(\mathbb{A}), \omega).$$

$\circlearrowleft T_\omega$ $G(\mathbb{A})$ $\qquad$ $\circlearrowleft T_\omega$ $C_c(G(\mathbb{A})) \ni f$

$$T_\omega(f)\varphi(x) := \int_{G(\mathbb{A})} f(y)\, T_\omega(y)\varphi(x)\, dy = \int_{G(\mathbb{A})} f(y)\, \varphi(xy)\, dy$$

<u>Thm</u>: $T_\omega(f)$ is a compact operator on $L_0^2(G(k)\backslash G(\mathbb{A}), \omega)$.

Key pt : estimate

$y \mapsto x^{-1}y$

$$T(f)\varphi(x) = \int_{G(A)} f(y)\varphi(xy)\,dy = \int_{G(A)} f(x^{-1}y)\varphi(y)\,dy$$

$$= \int_{G(k)\backslash G(A)} \sum_{\xi \in G(k)} f(x^{-1}\xi y)\varphi(y)\,dy$$

$$\left( \begin{aligned} K_f(x,y) &:= \boxed{\sum_{\xi \in G(k)}} f(x^{-1}\xi y) \leftarrow \text{bad choice } L_0^2 \\ K_f'(x,y) &= K_f(x,y) - \boxed{\int_{U(k)\backslash U(A)}} K_f(x,uy)\,du. \end{aligned} \right)$$

$$K_f(x,y) = \sum_{\xi \in U(k)} f(x^{-1}\xi y) - \int_{U(A)} f(x^{-1}uy)\,du$$

$$T(f)\varphi(x) = \int_{U(k)\backslash G(A)} K_f(x,y)\varphi(y)\,dy$$

$y \mapsto u^{-1}y$

$$\int_{U(k)\backslash G(A)} \int_{U(A)} f(x^{-1}uy)\varphi(y)\,du\,dy = \int_{G(A)} \int_{U(A)/U(k)} f(x^{-1}uy)\varphi(y)\,du\,dy$$

$$= \int_{G(A)} \int_{U(k)\backslash U(A)} f(x^{-1}y)\varphi(u^{-1}y)\,du\,dy = 0$$

Estimate $K_f(x,y)$

$f_{x,y}(u) := f(x^{-1}uy)$

$u \in U(\mathbb{A}) \xrightarrow{b} \mathbb{A}$

$$\sum_{\xi \in U(k)} f(x^{-1}\xi y) = \sum_{\xi \in U(k)} \hat{f}_{x,y}(\xi)$$ Poisson summation formula

$$(u,v) := \tau(b(u)b(v))$$ $\begin{pmatrix} 1 & b \\ & 1 \end{pmatrix} = u$

$$\hat{f}_{x,y}(v) := \int_{U(\mathbb{A})} f_{x,y}(u)(u,v)\,du$$

Iwasawa

$x = u_x h_x k_x$

$y = u_y h_y k_y$

$U(\mathbb{A})H(\mathbb{A})K$

$\begin{pmatrix} * & \\ & * \end{pmatrix}$

$\cup$

$\left\{\begin{pmatrix} a & \\ & a \end{pmatrix}\right\} = Z(\mathbb{A})$

$$K_f(x,y) = \sum_{\xi \in U(k)\setminus\{Id\}} \int_{U(\mathbb{A})} f_{x,y}(u)(u,\xi)\,du$$

$u \mapsto u_x u u_y^{-1}$

$$= \sum_{\xi} \int_{U(\mathbb{A})} f(k_x^{-1}h_x^{-1}\cancel{u_x^{-1}}\,u\,\cancel{u_y}\,h_y k_y)\,(\overset{u_x \quad u_y^{-1}}{u},\xi)\,du$$

$u \mapsto h_x u h_x^{-1}$

this will produce the modular char $\beta$ from $du$

$$= \sum_{\xi} (u_x u_y^{-1},\xi) \int_{U(\mathbb{A})} f(k_x^{-1}\,u\,h_x^{-1}h_y k_y)(h_x u h_x^{-1},\xi)\,|\beta(h_x)|\,du$$

$$\begin{pmatrix} 1 & b \\ & 1 \end{pmatrix}\begin{pmatrix} t_{x,1} & \\ & t_{x,2} \end{pmatrix}^{-1}\begin{pmatrix} t_{y,1} & \\ & t_{y,2} \end{pmatrix} = \begin{pmatrix} t_{x,1}^{-1}t_{y,1} & * \\ & t_{x,2}^{-1}t_{y,2} \end{pmatrix}$$

$f$ cpt supp $\Rightarrow$ $u h_x^{-1} h_y$ must be in some cpt in order to contribute.

$\Rightarrow h_x^{-1}h_y$ in a certain cpt, denote it by $\Omega_H$ $\quad \Omega_H \subset H(\mathbb{A})$

$$\beta\left(\begin{pmatrix} t_1 & \\ & t_2 \end{pmatrix}\right) = t_1 t_2^{-1}$$

$$F_{x,y}(u) := f(k_x^{-1} u h_x^{-1} h_y k_y)$$ "cpt family" as $x, y$ vary

$$\hat{F}_{x,y}(v) = \int_{U(\mathbb{A})} f(k_x^{-1} u h_x^{-1} h_y k_y)(u, v)\, du$$

$\Rightarrow$ a "cpt family" in Schwartz space

$$|K_f(x,y)| \leq \sum_{\xi \in U(k) \setminus \{Id\}} |\hat{F}_{x,y}(h_x \xi h_x^{-1})|\, |\beta(h_x)|$$

$$< \sum_{\xi} F(h_x \xi h_x^{-1})\, |\beta(h_x)|$$

$\exists$ $F$ Schwartz fun that bounds all $\hat{F}_{x,y}$

$$< C_N |\beta(h_x)|^{-N} \quad \forall N \in \mathbb{Z}_{>0}.$$

for $x \in \mathfrak{S}$ ( so $|\beta(h_x)| > c$ )

$$|T(f)\varphi(x)| = \left| \int_{\underbrace{U(k)\backslash G(\mathbb{A})}} K_f(x,y)\varphi(y)\, dy \right| < C_N' |\beta(h_x)|^{-N} \int_{Z(\mathbb{A})\backslash \mathfrak{S}} |\varphi(y)|\, dy$$

only this part can contribute $\to$ $U(k)\backslash U(\mathbb{A}) \cdot h_x \Omega_H K$

$h_x^{-1} h_y \in \Omega_H$

$$< C_N'' |\beta(h_x)|^{-N} \|\varphi\|_2, \forall N$$

arguments omitted that shows $T(f)$ is Hilbert-Schmidt integral operator.

$\Rightarrow$ $T(f)$ is a cpt op.

Ref: Godement Analyse Spectrale des fonctions modulaires.

$\square$

$\Longrightarrow$ ( See page 3.6, somebody may give a talk on deducing Cor from thm. later )

<u>Cor</u>: $L^2_0(G(k)\backslash G(\mathbb{A}), \omega) = \widehat{\bigoplus_{\pi}} \mathcal{H}_\pi$ top. direct sum

$\nwarrow$ countably many terms

$(\pi, \mathcal{H}_\pi)$ (top.) irred. subreps of $G(\mathbb{A}) \subseteq L^2_0(G(k)\backslash G(\mathbb{A}), \omega)$

$m_\pi := \#\{ (\pi', \mathcal{H}_{\pi'}) \subseteq L^2_0(G(k)\backslash G(\mathbb{A}), \omega) \mid \pi' \simeq \pi$ as (abstract) $G(\mathbb{A})$-repn $\}$

$m_\pi < \infty$

( will show, in fact, $m_\pi = 1$ )